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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1345

TRANSLATIONAL MOTION OF BODIES UNDER THE FREE SURFACE OF A HEAVY FLUID OF FINITE DEPTH

By M. D. Haskind

Translation

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TRANSLATIONAL MOTION OF BODIES UNDER THE FREE SURFACE OF A HEAVY FLUID OF FINITE DEPTH*

By M. D. Haskind

In reference 1, entitled "The Two-Dimensional Problem of the Vibration of Bodies under the Surface of a Heavy Fluid of Finite Depth," the problem was to determine the wave motion of a heavy fluid excited by the periodic vibrations of a body of arbitrary shape situated under the free surface of the fluid of finite depth; the method of N. E. Kochin (reference 2) was used.

In the present paper, the two-dimensional problem of the wave motion produced in a heavy fluid of finite depth by the horizontal rectilinear and uniform motion of a solid body of arbitrary shape immersed under the surface of the fluid is considered by the same method.

1. Statement of the Problem

The problem of the translatory motion of a solid body under the free surface of a heavy incompressible fluid of finite depth will be considered. The case in which the motion of the body occurs with constant horizontal velocity c will be studied. The motion of the fluid will be defined with reference to a moving system of coordinates Oxy fixed to the body, the x-axis coinciding with the undisturbed level of the fluid and directed along the direction of motion of the body, and the y-axis directed vertically upward.

It will be assumed that the motion of the fluid is potential and steady relative to the body. From the integral of Lagrange for the pressure within the fluid,

$$p - p_0 = \rho c \frac{\partial \varphi}{\partial x} - \rho \frac{1}{2} v^2 - \rho g y \tag{1.1}$$

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where p_0 is the atmospheric pressure, ρ the density of the fluid, g the acceleration of gravity; $\varphi(x,y)$ the potential of the absolute motion of the fluid, and $v = |\text{grad } \varphi|$ the magnitude of the absolute velocity of the fluid.

The function $\varphi(x,y)$ is determined from the boundary conditions; the flow condition on the wetted contour of the body,

$$\frac{\partial\varphi}{\partial n} = \dot{c} \cos(n,x) \quad \text{on } C \tag{1.2}$$

where n is the outer normal to the contour C;
on the free boundary $p = p_0$, and hence

$$c \frac{\partial\varphi}{\partial x} - \frac{1}{2} v^2 - gy = C \tag{1.3}$$

on the bottom of the channel for $y = -h_0$, the following condition applies

$$\frac{\partial\varphi}{\partial y} = 0 \tag{1.4}$$

According to the theory of waves of small amplitude, condition (1.3) may be linearized. For this purpose the boundary condition (1.3) is referred to the x-axis and the term $v^2/2$ neglected. In place of condition (1.3),

$$\frac{\partial\varphi}{\partial x} - \frac{g}{c} y(x) = 0 \tag{1.5}$$

It is easily seen that on the free surface the following relation holds

$$cy(x) = \varphi + \text{const} \tag{1.6}$$

where ψ is the stream function. In fact, when the stream function of the motion of the fluid relative to the body is denoted by ψ_0, there is obtained

$$\psi_0 = \psi - cy$$

or

$$cy = \psi - \psi_0$$

From this relation, equation (1.6) follows, since the boundary of the fluid in the relative motion is represented by stream lines on which ψ_0 is constant. For the free surface, it may be assumed that $\psi_0 = 0$. Hence, on the free surface,

$$cy(x) = \psi$$

and therefore boundary condition (1.5) assumes the form

$$\frac{\partial\varphi}{\partial x} - \nu\psi = 0 \qquad \text{for} \quad y = 0 \tag{1.7}$$

where

$$\nu = \frac{g}{c^2} \tag{1.8}$$

From condition (1.5) it is seen that the equation of the free surface will be

$$y(x) = \frac{c}{g}\left[\frac{\partial\nu}{\partial x}\right]_{y=0} \tag{1.9}$$

2. Fundamental Formulas of the Problem

The problem may be mathematically formulated as follows. It is required to determine the characteristic function $w(z) = \varphi + i\psi$ $(z = x + iy;\ i = \sqrt{-1})$, satisfying the conditions:

1. For $0 > y > -h_0$ in the region occupied by the fluid, the derivative dw/dz is finite and at infinity for $x \to +\infty$, the derivative dw/dz vanishes.

2. On the contour C, the smooth flow condition applies

$$\frac{\partial\varphi}{\partial n} = c \cos(n,x)$$

3. On the free surface for $y = 0$, the linearized condition holds with regard to the constancy of the pressure

$$\mathrm{Re}(dw/dz + i\nu w) = 0$$

4. On the bottom of the channel for $y = -h_0$, the following condition holds

$$\operatorname{Im} dw/dz = 0$$

In the region occupied by the fluid, the point z is taken and two contours C_1 and C_∞ are drawn, of which C_∞ contains both the point z and the contour C, while the C_1 contains the contour C, but not the point z (fig. 1). By the formula of Cauchy for a single-valued function $dw/dz = \overline{v}(z)$,

$$\overline{v}(z) = \frac{1}{2\pi i}\int_{C_1} \frac{\overline{v}(\zeta)\,d\zeta}{z - \zeta} - \frac{1}{2\pi i}\int_{C_\infty} \frac{v(\zeta)\,d\zeta}{z - \zeta} \tag{2.1}$$

where the bar over a letter indicates, as usual, the transition to the complex conjugate value. The following notation is introduced

$$V_1(z) = \frac{1}{2\pi i}\int_{C_1} \frac{\overline{v}(\zeta)\,d\zeta}{z - \zeta} \qquad V_2(z) = -\frac{1}{2\pi i}\int_{C_\infty} \frac{v(\zeta)\,d\zeta}{z - \zeta} \tag{2.2}$$

It is evident that $V_1(z)$ is a holomorphic function in the entire plane of the complex variable outside the contour C_1, having at infinity the order z^{-1} and capable of being continued analytically in the entire part of the complex variable plane which lies outside the contour C, while $V_2(z)$ is a holomorphic function within the contour C_∞, by the extension of which an analytical continuation of this function may be obtained over the entire strip $0 > y > -h_0$.

The function $V_2(z)$ may be represented in another form. For this it is possible to find a function $\omega(z)$, which in the strip $0 > y > -h_0$ has a single pole of the first order $\zeta = \xi + i\eta$ with residue $A/2\pi i$ and which satisfies conditions 1, 3, and 4.

In fact, for a vortex of strength Γ, located at the complex point $\zeta = \xi + i\eta$, an expression for the complex velocity was obtained by Tikhonov (reference 3)

$$\omega_{\Gamma}(z) = \frac{\Gamma}{2\pi i(z - \zeta)} - \frac{\Gamma}{2\pi i(z - \bar{\zeta} + 2ih_0)} -$$

$$\frac{\Gamma}{\pi}\int_0^{\infty} (\nu + \lambda)\ \exp(-\ \lambda h_0)\ \frac{\operatorname{sh} \lambda(\eta + h_0)\ \cos \lambda(z - \xi + ih_0)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0}\ d\lambda -$$

$$\Gamma_{\nu}\ \frac{\operatorname{sh} \lambda_0(\eta + h_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0}\ \sin\ _0(z - \xi + ih_0) \tag{2.3}$$

where λ_0 is the real and positive root of the equation

$$\nu \operatorname{sh} \lambda h_0 = \lambda \operatorname{ch} \lambda h_0 \tag{2.4}$$

For $c^2 < \sqrt{gh_0}$, in all cases where the function to be integrated has a singularity, the principal value in the sense of Cauchy is taken under the integral.

For $c^2 > \sqrt{gh_0}$, equation (2.4) has only imaginary roots and the fourth term of formula (2.3), which determines the presence of free waves, is absent.

For a source of strength Q located at the complex point $\zeta = \xi + i\eta$, the expression of the complex velocity may be obtained in the same manner as in the case of a vortex. Without the computations, the final result is

$$\omega_Q(z) = \frac{Q}{2\pi(z - \zeta)} + \frac{Q}{2\pi(z - \bar{\zeta} + 2ih_0)} +$$

$$\frac{Q}{\pi}\int_0^{\infty} (\nu + \lambda)\ \exp(-\ \lambda h_0)\ \frac{\operatorname{ch} \lambda(\eta + h_0)\ \sin \lambda(z - \xi + ih_0)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0}\ d\lambda -$$

$$Q\nu\ \frac{\operatorname{ch} \lambda_0(\eta + h_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0}\ \cos \lambda_0(z - \xi + ih_0) \tag{2.5}$$

By the use of expressions (2.3) and (2.5), to obtain the function $\omega(z)$ may be obtained without difficulty. For this purpose, since $A = \Gamma + iQ$, the following expression is obtained after simple transformations:

$$\varpi(z) = \frac{A}{2\pi i(z - \zeta)} - \frac{\bar{A}}{2\pi i(z - \bar{\zeta} + 2ih_0)} -$$

$$\frac{1}{2\pi i}\int_0^{\infty} (\nu + \lambda)\exp(-\lambda h_0)\frac{\bar{A}\sin\lambda(z - \bar{\zeta} + 2ih_0) - A\sin\lambda(z - \zeta)}{\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda h_0}\,d\lambda +$$

$$\frac{\nu}{2i}\,\frac{\bar{A}\cos\lambda_0(z - \bar{\zeta} + 2ih_0) - A\cos\lambda_0(z - \zeta)}{\nu h_0 - \mathrm{ch}^2\,\lambda_0 h_0} \tag{2.6}$$

Here, as in the preceding formulas, the fourth term, which determines the presence of free waves, is present only if $c^2 < \sqrt{gh_0}$.

When $A = \bar{v}(\zeta)\,d\zeta$ is substituted in the previous formula and integration is carried out over the contour C_1,

$$v(z) = V_1(z) - \frac{1}{2\pi i}\int_{C_1} v(\zeta)\left\{\frac{1}{z - \bar{\zeta} + 2ih_0} + \int_0^{\infty}(\nu + \lambda)\exp(-\lambda h_0)\frac{\sin\lambda(z - \bar{\zeta} + 2ih_0)}{\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda h_0}\,d\lambda - \right.$$

$$\left.\pi\nu\,\frac{\cos\lambda_0(z - \bar{\zeta} + 2ih_0)}{\nu h_0 - \mathrm{ch}^2\,\lambda_0 h_0}\right\}d\bar{\zeta} + \frac{1}{2\pi i}\int_{C_1}\bar{v}(\zeta)\left\{\int_0^{\infty}(\nu + \lambda)\exp(-\lambda h_0)\,\frac{\sin\lambda(z - \zeta)}{\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda h_0}\,d\lambda - \right.$$

$$\left.\pi\nu\,\frac{\cos\lambda_0(z - \zeta)}{\nu h_0 - \mathrm{ch}^2\,\lambda_0 h_0}\right\}d\zeta \tag{2.7}$$

If both points z and ζ are situated in the strip $0 > y > -h_0$, the following equation holds

$$\frac{1}{z - \bar{\zeta} + 2ih_0} = -i\int_0^{\infty}\exp\left[i\lambda(z - \bar{\zeta} + 2ih_0)\right]d\lambda \tag{2.8}$$

With this equation taken in account, it is found from equation (2.7) that the function $V_2(z)$ can be represented in the form

$$V_2(z) = -\frac{1}{2\pi i}\int_{C_1} v(z)\left\{\int_0^{\infty}\left[-i\exp\left[i\lambda(z-\bar{\zeta}+2ih_0)\right]+\right.\right.$$

$$\left.\left.(\nu+\lambda)\exp(-\lambda h_0)\frac{\sin\lambda(z-\bar{\zeta}+2ih_0)}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\right]d\lambda-\pi\nu\frac{\cos\lambda_0(z-\bar{\zeta}+2ih_0)}{\nu h_0-\operatorname{ch}^2\lambda_0 h_0}\right\}d\bar{\zeta}+$$

$$\frac{1}{2\pi i}\int_{C_1}\bar{v}(\zeta)\left\{\int_0^{\infty}(\nu+\lambda)\exp(-\lambda h_0)\frac{\sin\lambda(z-\zeta)}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}d\lambda-\pi\nu\frac{\cos\lambda_0(z-\zeta)}{\nu h_0-\operatorname{ch}^2\lambda_0 h_0}\right\}d\zeta \tag{2.9}$$

The conjugate complex functions are introduced for real λ

$$H(\lambda)=\int_{C_1}\bar{v}(\zeta)\exp - i\lambda\zeta d\zeta,\qquad \bar{H}(\lambda)=\int_{C_1}v(\zeta)\exp i\lambda\bar{\zeta}d\bar{\zeta} \tag{2.10}$$

By an interchange in equation (2.9) of the order of integration, and by simple transformations, there is readily obtained

$$V_2(z)=\frac{1}{2\pi}\left\{\int_0^{\infty}\left[\bar{H}(-\lambda)\exp\left[i\lambda(z+2ih_0)\right]+\right.\right.$$

$$\frac{(\nu+\lambda)\exp(-\lambda h_0)}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\Bigg(\bar{H}(-\lambda)\exp\left[i\lambda(z+2ih_0)\right]-\bar{H}(\lambda)\exp\left[-i\lambda(z+2ih_0)\right]-$$

$$\left.H(\lambda)\exp i\lambda z+H(-\lambda)\exp(-i\lambda z)\Bigg)\right]d\lambda-$$

$$\frac{\pi i\nu}{2(\nu h-\operatorname{ch}^2\lambda_0 h_0)}\Bigg(\bar{H}(-\lambda_0)\exp i\lambda_0(z+2ih_0)+\bar{H}(\lambda_0)\exp\left[-i\lambda_0(z+2ih_0)\right]-$$

$$\left.H(\lambda_0)\exp i\lambda_0 z-H(-\lambda_0)\exp(-i\lambda_0 z)\Bigg)\right\} \tag{2.11}$$

It is of interest to find the character of the waves that remain behind the moving body. For this purpose the asymptotic expression of the complex velocity is first obtained for $x\to-\infty$ in the case of a vortex and source. In reference 3, the asymptotic expression of the complex velocity in the case of a vortex is of the form

$$(\omega_\Gamma)_{x \to -\infty} = - 2\Gamma\nu \frac{\operatorname{sh} \lambda_0(\eta + h_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \sin \lambda_0(z - \xi + ih_0) \tag{2.12}$$

In a similar manner, the asymptotic expression of the complex velocity is obtained in the case of a source. Without the computations, the final result is

$$(\omega_Q)_{x \to -\infty} = - 2Q\nu \frac{\operatorname{ch} \lambda_0(\eta + h_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \cos \lambda_0(z - \xi + ih_0) \tag{2.13}$$

For the function $\omega(z)$, having a polarity with residue $A/2\pi i$, the following asymptotic expression is obtained:

$$(\omega)_{x \to -\infty} = - i\nu \frac{\overline{A} \cos \lambda_0(z - \overline{\zeta} + 2ih_0) - A \cos \lambda_0(z - \zeta)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \tag{2.14}$$

Setting $A = \overline{v}(\zeta)\ d\zeta$ and integrating over the contour C_1 yields the asymptotic expression of the function $\overline{v}(z) = dw/dz$:

$$\left(\frac{dw}{dz}\right)_{x \to -\infty} = - \frac{i\nu}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} \Big[\overline{H}(- \lambda_0) \exp i\lambda_0(z + 2ih_0) +$$

$$\overline{H}(\lambda_0) \exp\left[- i\lambda_0(z + 2ih_0)\right] - H(\lambda_0) \exp i\lambda_0 z - H(- \lambda_0) \exp(- i\lambda_0 z)\Big] \tag{2.15}$$

Finally, from the formula

$$y(x) = \frac{c}{g} \operatorname{Re} \left(\frac{dw}{dz}\right)_{y=0}$$

it is readily found that for $x \to -\infty$ sinusoidal waves of length $2\pi/\lambda_0$ are formed behind the amplitude of which, after some simple transformations, may be represented in the form

$$a = \frac{\operatorname{ch} \lambda_0 h_0}{c(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} \left| \overline{H}(\lambda_0) \exp \lambda_0 h_0 - H(- \lambda_0) \exp - \lambda_0 h_0 \right| \tag{2.16}$$

3. Formulas for Determining the Forces

The forces acting on the contour C are now computed. The lift force of the contour is denoted by P, the resistance by R, and the moment of the forces on the contour about the origin by M. These forces will be computed by the formulas of Chaplygin-Blasius:

$$P - iR = -\frac{\rho}{2}\int_{C_2} \overline{v}^2{}_0(z)\, dz, \qquad M = \operatorname{Re}\frac{\rho}{2}\int_{C_2} z\overline{v}^2{}_0(z)\, dz \tag{3.1}$$

where C_2 is an arbitrary contour, situated in the region $0 > y > -h_0$ and containing the contour C; and $\overline{v}_0(z)$ is the complex velocity in the relative motion obtained by superposing on the absolute flow a uniform motion of the fluid with velocity c in the direction of the negative x-axis. Thus,

$$\overline{v}_0(z) = V_1(z) + V_2(z) - c$$

where the contour C_1 is chosen to lie between C and C_2.

Formulas (3.1) do not take into account the buoyancy force of Archimedes, equal to $g\rho S$, and it moment, equal to $-g\rho Sx_c$, where S is the area that bounds the contour C, and x_c is the abscissa of the center of gravity of this area.

The following integral is now computed:

$$J = \int_{C_2} \overline{v}_0{}^2(z)\, dz = \int_{C_2} V_1{}^2(z)\, dz + \int_{C_2} (V_2(z) - c)^2\, dz + 2\int_{C_2} V_1(V_2 - c)\, dz$$

But the first and second integrals on the right are equal to zero because the function $V_1(z)$ is holomorphic outside the contour C_2 and has at infinity a zero of at least the first order, while the function $V_2(z)$ is holomorphic within the contour C_2. Hence,

$$J = 2\int_{C_2} V_1V_2\, dz - 2c\int_{C_2} (V_1 + V_2)\, dz = 2\int_{C_2} V_1V_2\, dz - 2c\int_{C_2} \overline{v}(z)\, dz$$

The velocity circulation about any contour that contains the contour C is denoted by Γ, so that

$$\Gamma = \int_C \overline{v}(z)\, dz$$

therefore

$$P - iR = -\rho \int_{C_2} V_1 V_2 \, dz + \rho c \Gamma \qquad (3.2)$$

By the use of expressions (2.2) and (2.11), the following expression is obtained

$$\int_{C_2} V_1(z)\, V_2(z)\, dz = \frac{1}{2\pi i} \int_{C_2} \frac{1}{2\pi} \int_{C_1} \frac{\overline{v}(\zeta)}{z - \zeta} \Biggl\{ \int_0^{\infty} \Biggl[\overline{H}(-\lambda) \exp i\lambda(z + 2ih_0) +$$

$$\frac{\nu + \lambda}{2} \exp(-\lambda h_0) \frac{\overline{H}(-\lambda) \exp i\lambda(z + 2ih_0) - \overline{H}(\lambda) \exp\left[-i\lambda(z + 2ih_0)\right]}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} -$$

$$\frac{\nu + \lambda}{2} \exp(-\lambda h_0) \frac{H(\lambda) \exp i\lambda z - H(-\lambda) \exp(-i\lambda z)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} \Biggr] d\lambda -$$

$$\frac{\pi i \nu}{2} \frac{\overline{H}(-\lambda_0) \exp i\lambda_0(z + 2ih_0) + \overline{H}(\lambda_0) \exp\left[-i\lambda_0(z + 2ih_0)\right]}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} +$$

$$\frac{\pi i \nu}{2} \frac{H(\lambda_0) \exp i\lambda_0 z + H(-\lambda_0) \exp(-i\lambda_0 z)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \Biggr\} d\zeta\, dz$$

Since the point ζ, which belongs to the contour C_1, lies within the contour C_2, with an interchange in the order of integration and by the following formula,

$$\frac{1}{2\pi i} \int_{C_2} \frac{e^{\pm i\lambda z}}{z - \zeta} dz = e^{\pm i\lambda\zeta}$$

There is obtained

$$\int_{C_2} V_1(z)\, V_2(z)\, dz = \frac{1}{2\pi}\left\{\int_0^{\infty}\left[|H(-\lambda)|^2 \exp(-2\lambda h_0) + \right.\right.$$

$$\left.\frac{\nu+\lambda}{2}\exp(-\lambda h_0)\,\frac{|H(-\lambda)|^2 \exp(-2\lambda h_0) - |H(\lambda)|^2 \exp 2\lambda h_0}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0}\right] d\lambda -$$

$$\left.\frac{\pi i\nu}{2}\,\frac{H(-\lambda_0)|^2 \exp(-2\lambda_0 h_0) + |H(\lambda_0)|^2 \exp 2\lambda_0 h_0 - 2H(\lambda_0)\,H(-\lambda_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0}\right\} \tag{3.3}$$

Hence, formula (2.3) assumes the form

$$P - iR = \rho c\Gamma - \frac{\rho}{2\pi}\int_0^{\infty}\left[|H(-\lambda)|^2 \exp(-2\lambda h_0) + \right.$$

$$\left.(\nu+\lambda)\exp(-\lambda h_0)\,\frac{|H(-\lambda)|^2 \exp(-2\lambda h_0) - |H(\lambda)|^2 \exp 2\lambda h_0}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)}\right] d\lambda +$$

$$\frac{i\nu\rho}{4}\,\frac{|H(-\lambda_0)|^2 \exp(-2\lambda_0 h_0) + |H(\lambda_0)|^2 \exp 2\lambda_0 h_0 - 2H(\lambda_0)\,H(-\lambda_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \tag{3.4}$$

Separating the real and imaginary parts and adding to P the Archimedes force, not taken into account by the Chaplygin-Blasius formula, results in

$$P = \rho c\Gamma - \frac{\rho}{2\pi}\int_0^{\infty}\left[|H(-\lambda)|^2 \exp(-2\lambda h_0) + (\nu+\lambda)\exp(-\lambda h_0) \times \right.$$

$$\left.\frac{|H(-\lambda)|^2 \exp(-2\lambda h_0) - |H(\lambda)|^2 \exp 2\lambda h_0}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)}\right] d\lambda + \nu\rho\,\frac{\operatorname{Im}\left\{H(\lambda_0)\,H(-\lambda_0)\right\}}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} + g\rho S \tag{3.5}$$

$$R = -\frac{\rho\nu}{4}\,\frac{|H(\lambda_0)|^2 \exp 2\lambda_0 h_0 + |H(-\lambda_0)|^2 \exp(-2\lambda_0 h_0) - 2\operatorname{Re}\left\{H(\lambda_0)\,H(-\lambda_0)\right\}}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \tag{3.6}$$

Formula (3.6) may be given another form, namely

$$R = -\frac{\rho\nu}{4}\frac{\left|\overline{H}(\lambda_0)\exp\lambda_0 h_0 - H(-\lambda_0)\exp(-\lambda_0 h_0)\right|^2}{\mathrm{ch}^2\lambda_0 h_0 - \nu h_0} \tag{3.7}$$

It can be readily shown that the total resistance of the underwater wing consists only of the wave resistance. In fact, by the following well-known formula for computing the wave resistance in the case of a fluid of finite depth,

$$R = \frac{1}{4}\rho g a^2\left(1 - \frac{2\lambda_0 h_0}{\mathrm{sh}\ 2\lambda_0 h_0}\right) \tag{3.8}$$

and with the value of the amplitude a from formula (2.16), formula (3.7) is obtained after some transformations.

The moment of the acting forces on the contour C is now computed. When the moment of the Archimedes force is taken into account,

$$M = -g\rho S x_c + \mathrm{Re}\,\frac{\rho}{2}\int_{C_1} z\left[V_1(z) + V_2(z) - c\right]^2 dz \tag{3.9}$$

This expression is computed in an entirely similar manner to the computation of the expression $P - iR$.

For very large absolute values of z the following expansion can be employed

$$V_1(z) = \frac{1}{2\pi i}\int_C \frac{\overline{v}(\zeta)\,d\zeta}{z - \zeta} = \frac{1}{2\pi i z}\int_{C_1}\overline{v}(\zeta)\,d\zeta + \ldots = \frac{\Gamma}{2\pi i z} + \ldots$$

and, hence,

$$\int_{C_2} z\,V_1{}^2(z)\,dz = \frac{\Gamma^2}{2\pi i}, \qquad \mathrm{Re}\int_{C_2} z\,V_1{}^2(z)\,dz = 0$$

Further,

$$\int_{C_2} z(V_3 - c)^2\, dz = 0$$

and therefore,

$$M = - g\rho Sx_c + \mathrm{Re}\rho \int_{C_2} z\, V_1(V_2 - c)\, dz$$

or, since the function $V_2(z)$ is holomorphic within the contour C_2

$$M = - g\rho Sx_c - \rho c\, \mathrm{Re} \int_{C_2} z\bar{v}(z)\, dz + \rho\, \mathrm{Re} \int_{C_2} z\, V_1(z)\, V_2(z)\, dz \tag{3.10}$$

It is noted that

$$H'(\lambda) = \frac{dH}{d\lambda} = - i \int_{C_2} \zeta\bar{v}(\zeta)\, \exp(- i\lambda\zeta)\, d\zeta$$

The integrals in formula (3.10) are computed in the same manner as in the expression (3.3), and as a result there is obtained the formula

$$\begin{aligned} M = &- g\rho Sx_c - \rho c\, \mathrm{Re}\left[iH'(0)\right] + \rho\, \mathrm{Re}\left\{\frac{1}{2\pi i}\int_0^\infty \Big[H'(-\lambda)\, \bar{H}(-\lambda)\, \exp(- 2\lambda_0 h_0) + \right. \\ &\frac{(\nu + \lambda)\exp(-\lambda h_0)}{2(\nu\, \mathrm{sh}\, \lambda h_0 - \lambda\, \mathrm{sh}\, \lambda h_0)}\Big(H'(-\lambda)\, H(-\bar{\lambda})\, \exp(- 2\lambda h_0) + H'(\lambda)\, \bar{H}(\lambda)\, \exp 2\lambda h_0 - \\ &\qquad H'(-\lambda)\, H(\lambda) - H(-\lambda)\, H'(\lambda)\Big)\Big]\, d\lambda - \\ &\frac{\nu}{4(\nu h_0 - ch^2\, \lambda_0 h_0)}\Big(H'(-\lambda_0)\, \bar{H}(-\lambda_0)\, \exp(- 2\lambda_0 h_0) - H'(\lambda_0)\, \bar{H}(\lambda_0)\, \exp 2\lambda_0 h_0 - \\ &\qquad \left. H'(-\lambda_0)\, H(\lambda_0) + H(-\lambda_0)\, H'(\lambda_0)\Big)\right\} \end{aligned} \tag{3.11}$$

Formulas (3.5), (3.7), and (3.11) in the limiting case for $h_0 \to \infty$ agree with the formulas obtained by Kochin in reference 2.

The function $H(\lambda)$ in formulas (3.5), (3.7), and (3.11) does not depend on the contour C_1, and for example, the contour C or some other contour which contains the contour C may be taken for the contour of integration. Moreover, the value of the function $H(\lambda)$ does not change if, instead of the complex velocity $\overline{v}(z)$ of the absolute motion, the complex velocity of the relative motion $\overline{v}_0(z)$ is taken, because these two functions differ by a constant c. The properties of the function $H(\lambda)$ will be used in the following section.

4. Examples

In the preceding sections expressions were found in terms of the function $H(\lambda)$ of a number of important magnitudes, namely, the amplitude of the waves formed, the wave resistance, the lift force, and the moment of the forces acting on the contour. Thus, the function

$$H(\lambda) = \int_C \overline{v}(\zeta) \exp(-i\lambda\zeta)\, d\zeta = \int_C dw \exp(-i\lambda\zeta) \tag{4.1}$$

plays a fundamental part for the problem under consideration. In order to compute this function, it is necessary to know the expression for the complex velocity, i.e., the solution of the hydrodynamic problem. In case the relative depth of the submerged contour C is sufficiently large, however, a good approximation is obtained if, in place of the function $\overline{v}(z)$, there is substituted in formula (4.1) the expression of the complex velocity which corresponds to the motion of the contour C in an infinite fluid.

Several examples of such an approximate solution of the problem will be considered

1. <u>The motion of a circular cylinder</u>. - The circular cylinder of radius b, situated at the depth h under the free surface of the fluid, is assumed to move with constant horizontal forward velocity c, since the circulation about the contour of the cylinder has a given value Γ. In this case, the characteristic function for the infinite fluid is known:

$$w(z) = -\frac{cb^2}{z + hi} + \frac{\Gamma}{2\pi i} \ln(z + ih)$$

Hence,

$$\bar{v}(z) = \frac{cb^2}{(z + hi)^2} + \frac{\Gamma}{2\pi i(z + ih)} \tag{4.2}$$

By formula (4.1) the function $H(\lambda)$ is now constructed:

$$H(\lambda) = \int_C \left[\frac{cb^2}{(z + hi)^2} + \frac{\Gamma}{2\pi i(z + hi)}\right] \exp - i\lambda z \, dz$$

Since the contour C contains one singular point $z = -ih$, there is obtained by the theorem on residues

$$H(\lambda) = (\Gamma + 2\pi cb^2\lambda) \exp - \lambda h \tag{4.3}$$

With the use of formula (3.7), the expression for the wave resistance of the cylinder is obtained

$$R = \rho\nu \frac{\left[\Gamma \operatorname{sh} \lambda_0(h_0 - h) + 2\pi c\lambda_0 b^2 \operatorname{ch} \lambda_0(h_0 - h)\right]^2}{\operatorname{ch}^2 \lambda_0 h_0 - \nu h_0} \tag{4.4}$$

and by the use of formula (3.5) the expression for the lift force of the cylinder is obtained

$$P = \rho c\Gamma - \frac{\rho\Gamma^2}{4\pi(h_0 - h)} + \frac{\rho cb^2\Gamma}{2(h_0 - h)^2} - \frac{\pi\rho c^2 b^4}{2(h_0 - h)^3} +$$

$$\frac{\rho}{2\pi}\int_0^\infty (\nu + \lambda) \exp(-\lambda h_0) \frac{(\Gamma^2 + 4\pi^2c^2b^4\lambda^2) \operatorname{sh} 2\lambda(h_0 - h) + 4\pi cb^2\Gamma\lambda \operatorname{ch} 2\lambda(h_0 - h)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} d\lambda + g\rho S \tag{4.5}$$

The integral component of this formula may be computed by the method of mechanical quadratures. In the limiting cases $\nu = 0$ and $\nu = \infty$, this component can be very accurately computed. Moreover, if this integral component is considered as a function of the parameter $\alpha = 1/(\nu h_0) = c^2/(gh_0)$, it can be shown that for $\alpha = 1$ this component suffers a discontinuity. In the particular case when the radius b of the cylinder is taken equal to zero, i.e., when the motion of a vortex under a free surface is considered, formulas (4.4) and (4.5) lead to the expressions established by Tikhonov. It is noted further that formulas (4.4) and (4.5) have been derived on the assumption that $c^2 < gh_0$. For $c^2 > gh_0$, no free waves are formed behind the cylinder and the wave resistance R is equal to zero.

For the moment of the forces exerted by the fluid on the cylinder, the following expression is obtained by formula (3.11):

$$M = -\frac{\rho\nu}{4}\frac{H'(-\lambda_0)\,H(-\lambda_0)\,\exp(-2\lambda_0 h_0) - H'(\lambda_0)\,H(\lambda_0)\,\exp 2\lambda_0 h_0}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0} +$$

$$\frac{\rho\nu}{4}\frac{H'(-\lambda_0)\,H(\lambda_0) - H(-\lambda_0)\,H'(\lambda_0)}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}$$

But from equation (4.3), it is evident that

$$H'(\lambda) = -hH(\lambda) + 2\pi cb^2 \exp(-\lambda h)$$

$$H'(-\lambda) = hH(-\lambda) - 2\pi cb^2 \exp \lambda h$$

Hence, after simple transformations,

$$M = hR - 2\pi\rho cb^2\nu\frac{\Gamma\,\operatorname{sh}^2\lambda_0(h_0 - h) + \pi cb^2\lambda_0\,\operatorname{sh} 2\lambda_0(h_0 - h)}{\operatorname{ch}^2\lambda_0 h_0 - \nu h_0} \tag{4.6}$$

The point of intersection with the y-axis of the resultant force on the body is determined by the formula

$$y_0 = -\frac{M}{R} = -h + \frac{2\pi cb^2}{\Gamma + 2\pi cb^2\lambda_0\,\operatorname{cth}\lambda_0(h_0 - h)} \tag{4.7}$$

It is evident that for $R > 0$ this resultant never passes through the center of the cylinder.

2. Motion of an elliptic cylinder. - An ellipse, having a center at the depth h and having axes 2α and 2β directed parallel to the axes of coordinates x and y, is allowed to move with a constant velocity c in the direction of the x-axis. The circulation Γ is, for simplicity, taken equal to zero. In this case, the flow of an infinite fluid about the contour C is determined with the aid of an auxiliary variable and the formula

$$z = -ih + \frac{1}{2}\sqrt{\alpha^2 - \beta^2}\left(u + \frac{1}{u}\right), \quad w = -\frac{c}{2}\sqrt{\alpha^2 - \beta^2}\left(u + \frac{r^2}{u}\right)$$

where $r = \sqrt{(\alpha + \beta)/(\alpha - \beta)}$ and $|u| = r$ is the equation of the circle in the u-plane which corresponds to the contour of the ellipse C. The exterior of this circle corresponds to the exterior of the ellipse. The following function is set up:

$$H(\lambda) = \int_C \exp(-\ i\lambda z)\ dw =$$

$$\int_{|u|=r} \left(-\ \frac{c}{2}\sqrt{\alpha^2 - \beta^2}\right)\left(1 - \frac{r^2}{u^2}\right) \exp\left[-\ \lambda h - \frac{i\lambda}{2}\sqrt{\alpha^2 - \beta^2}\left(u + \frac{1}{u}\right)\right] du$$

When the substitution $u = iv$ is made, there is obtained

$$H(\lambda) = -\ \frac{ic}{2}\sqrt{\alpha^2 - \beta^2}\ \exp(-\ \lambda h) \int_{|v|=r} \left(1 + \frac{r^2}{v^2}\right) \exp \frac{\lambda}{2}\sqrt{\alpha^2 - \beta^2}\left(v - \frac{1}{v}\right) dv$$

But by the theory of Bessel functions it is known that

$$\frac{1}{2\pi i}\int_{|v|=r} \frac{dv}{v^{n+1}} \exp \frac{z}{2}\left(v - \frac{1}{v}\right) = J_n(z)$$

hence,

$$H(\lambda) = \pi c\sqrt{\alpha^2 - \beta^2}\ \exp(-\ \lambda h)\left\{J_{-1}\left(\lambda\sqrt{\alpha^2 - \beta^2}\right) + r^2 J_1\left(\lambda\sqrt{\alpha^2 - \beta^2}\right)\right\}$$

From the formula

$$J_{-1}(z) = -\ J_1(z)$$

and the value of r, the following expression is obtained

$$H(\lambda) = 2\pi c\beta\sqrt{\frac{\alpha + \beta}{\alpha - \beta}}\ \exp(-\ \lambda h)\ J_1\left(\lambda\sqrt{\alpha^2 - \beta^2}\right) \tag{4.8}$$

The computation is restricted to the wave resistance. By formula (3.7),

$$R = 4\pi^2 \rho g \beta^2 \frac{\alpha + \beta}{\alpha - \beta} \frac{\operatorname{ch}^2 \lambda_0 (h_0 - h)}{\operatorname{ch}^2 \lambda_0 h_0 - \nu h_0} J_1{}^2\left(\lambda_0 \sqrt{\alpha^2 - \beta^2}\right) \tag{4.9}$$

From this formula, it follows that for certain λ_0 and, therefore, for a certain velocity $c < \sqrt{gh_0}$, the wave resistance is equal to zero; i.e., the amplitude of the waves formed behind the moving body becomes zero. This will be the case if the following relation is satisfied:

$$\lambda_0 \sqrt{\alpha^2 - \beta^2} = s_k \qquad (k=1,2, \ldots)$$

where s_k is the positive root of the Bessel function $J_1(s)$. The first root of this function is

$$s_1 = 3.832$$

Since the parameter $\nu = g/c^2$ is connected with λ_0 by the equation

$$\operatorname{th} \lambda_0 h_0 = \frac{c^2}{g} \lambda_0$$

the first velocity at which the wave resistance becomes zero is determined by the formula

$$c = 0.51 \sqrt{g\sqrt{\alpha^2 - \beta^2} \operatorname{th} \frac{3.832 h_0}{\sqrt{\alpha^2 - \beta^2}}} \tag{4.10}$$

Moreover,

$$\operatorname{th} \frac{3.832 h_0}{\sqrt{\alpha^2 - \beta^2}} < 1$$

hence,

$$c < 0.51 \sqrt{g\sqrt{\alpha^2 - \beta^2}} \tag{4.11}$$

In a similar manner a number of other examples may be considered. Moreover, as in reference 2, it is possible in this case to set up a functional equation for determining the function $H(\lambda)$ and the values of the circulation Γ from the condition of the finite velocity at the sharp edge. These equations may be obtained by the same method. Their final form will be somewhat more complicated as compared with the case of the infinite fluid.

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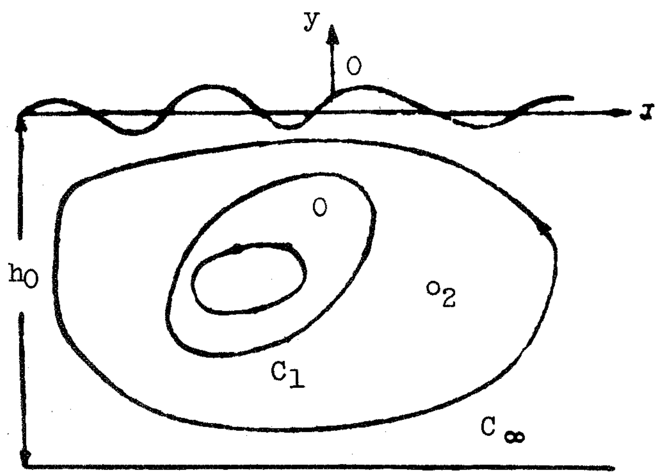

Figure 1.